SET THEORY HOMEWORK 6

Due Friday, May 5.

Problem 1. *If $V \subset W$ are two models of set theory and $V \models$ "D is club in κ", then $W \models$ "D is club in κ".*

Problem 2. *Let κ be a regular uncountable cardinal and $\mathbb{P}$ be a κ-c.c. poset, i.e. every antichain has size less than κ.*

(a) Suppose that $\dot{C}$ is a $\mathbb{P}$- name such that $1 \Vdash$ "$\dot{C}$ is a club subset of κ", and define $D = \{\alpha < \kappa \mid 1 \Vdash \alpha \in \dot{C}\}$. Show that D is a club a subset κ. (Note that D is in the ground model.)

(b) Show that $\mathbb{P}$ preserves stationary subsets of κ, i.e. if $S \subset \kappa$ is stationary in the ground model, then S remains stationary in any $\mathbb{P}$-generic extension.

Problem 3. *Let κ be a regular uncountable cardinal and $\mathbb{P}$ be a $< \kappa$-closed poset. Show that $\mathbb{P}$ preserves stationary subsets of κ.*
Hint: If $S \subset \kappa$ is stationary and $p \Vdash$ "$\dot{C}$ is a club subset of κ", show there is a sequence in the ground model $\langle p_\alpha, \gamma_\alpha \mid \alpha < \kappa\rangle$, such that:

- *$\langle p_\alpha \mid \alpha < \kappa\rangle$ is a decreasing sequence below p,*
- *$\langle \gamma_\alpha \mid \alpha < \kappa\rangle$ is a club in κ,*
- *each $p_\alpha \Vdash \gamma_\alpha \in \dot{C}$.*

Then use stationarity of S in the ground model.

Problem 4. *Let $S \subset \omega_1$ be a stationary set. Define $\mathbb{P} := \{p \subset S \mid p$ is closed and bounded$\}$, and set $p \leq q$ if p end extends q i.e. for some α, $p \cap \alpha = q$. Suppose that $T := S \setminus \omega_1$ is also stationary. Let G be a $\mathbb{P}$-generic filter. Show that in $V[G]$, T is nonstationary.*

The above is an example of a forcing that destroys a stationary set, without collapsing cardinals (the cardinal preservation takes an argument). On the other hand, you cannot destroy a club set (see problem 1).

Problem 5. *Suppose that, in V, S is a stationary subset of ω_1 and $\mathbb{P}$ is a forcing that destroys the stationarity of S. Show that there are ω_1-many dense subsets of $\mathbb{P}$, such that no $\mathbb{P}$-filter $G \in V$ meets them.*

The above problem is why Martin's Maximum is in a sense the optimal generalization of MA.